Crash Course: Continuous-Time LTI Systems

Control Systems Engineering

Introduction

This document provides a concise summary of the core concepts for analyzing and designing controllers for **Continuous-Time Linear Time-Invariant (CT LTI)** systems. The fundamental principle is that to *control* a system, you must first be able to *model* and *analyze* it.

1 Part 1: System Representation (The "What")

How do we mathematically describe a CT LTI system?

1.1 Differential Equation Model

The most fundamental form.

$$a_n \frac{d^n}{dt^n} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_m \frac{d^m}{dt^m} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t) \tag{1}$$

Where $x(t)$ is the input and $y(t)$ is the output.

1.2 Transfer Function Model, $H(s)$

Found by taking the **Laplace Transform** of the differential equation, assuming **zero initial conditions**.

$$H(s) = \frac{\mathcal{L}\{\text{output}\}}{\mathcal{L}\{\text{input}\}} = \frac{Y(s)}{X(s)} \tag{2}$$

It is a ratio of polynomials in the complex frequency variable s.
Key Takeaway: The transfer function provides a complete input-output description of the system's dynamics.

1.3 State-Space Model

Represents an n^{th}-order system as n coupled **first-order** differential equations.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \quad \text{(State Equation)} \tag{3}$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \quad \text{(Output Equation)} \tag{4}$$

Where:

- $\mathbf{x}(t) \in \mathbb{R}^n$: **State vector** (internal variables).
- $u(t) \in \mathbb{R}$: Input.

- $y(t) \in \mathbb{R}$: Output.
- $\mathbf{A} \in \mathbb{R}^{n \times n}$: **System matrix** (dynamics).
- $\mathbf{B} \in \mathbb{R}^{n \times 1}$: **Input matrix**.
- $\mathbf{C} \in \mathbb{R}^{1 \times n}$: **Output matrix**.
- $\mathbf{D} \in \mathbb{R}$: **Feedthrough matrix**.

2 Part 2: System Analysis (The "How is it behaving?")

Once we have a model, we analyze its properties and behavior.

2.1 Poles and Zeros

- **Poles:** Roots of the **denominator** of $H(s)$. They determine the **natural response** and **stability**.
- **Zeros:** Roots of the **numerator** of $H(s)$. They affect the **amplitude** and shape of the response.

2.2 Impulse Response, $h(t)$

The output of a system when the input is a Dirac delta function $\delta(t)$.

$$h(t) = \mathcal{L}^{-1}\{H(s)\} \tag{5}$$

Crucial Fact: For an LTI system, the output for *any* input is the convolution of the input with the impulse response:

$$y(t) = x(t) * h(t) \tag{6}$$

2.3 Stability

- **BIBO (Bounded-Input-Bounded-Output) Stability:** A system is stable if every bounded input produces a bounded output.
- **Test via Poles:** A CT LTI system is **BIBO stable if and only if all poles have negative real parts** (i.e., all poles are in the **left-half plane (LHP)** of the complex s-plane).

2.4 System Response to Standard Inputs

Analyze the **Step Response** (output for a unit step $u(t)$) to find key performance parameters:

- **Rise Time (T_r):** Speed of response.
- **Settling Time (T_s):** Time to reach and stay within a tolerance band (e.g., 2%).
- **Overshoot (M_p):** Maximum peak above the final value, expressed as a percentage.
- **Steady-State Error (e_{ss}):** Difference between desired and actual output as $t \to \infty$.

2.5 Frequency Response

How the system responds to sinusoidal inputs of different frequencies.

- **Found by evaluating $H(s)$ at $s = j\omega$:** $H(j\omega)$.
- Plotted as **Bode Plots** (Magnitude vs. Frequency and Phase vs. Frequency) or **Nyquist Plots**.

3 Part 3: Feedback Control Design (The "How to make it behave better")

We use feedback to alter the system's natural dynamics to meet performance specifications.

3.1 The Basic Feedback Loop

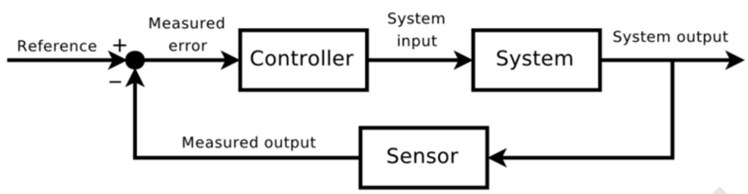

Figure 1: Basic closed-loop feedback system.

- **Plant ($G(s)$):** The system to be controlled.
- **Controller ($C(s)$):** The device we design.
- **Feedback Path ($H(s)$):** Often a sensor ($H(s) = 1$ for simplicity).
- **Closed-Loop Transfer Function (CLTF):**

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} \tag{7}$$

- **Characteristic Equation:**

$$1 + C(s)G(s)H(s) = 0 \tag{8}$$

  The roots of this equation are the **closed-loop poles**.

3.2 The Power of Feedback

Feedback can:

- **Stabilize** an unstable system.
- **Reduce** sensitivity to parameter variations.
- **Improve** disturbance rejection.
- **Speed up** slow responses and **manage** overshoot.
- **Reduce** steady-state error.

3.3 Root Locus Method

A graphical technique showing how the **closed-loop poles** move in the s-plane as a single parameter (typically controller gain, K) is varied from 0 to ∞.
Primary Use: For designing gain and understanding the trade-off between stability and response speed.

3.4 Frequency Domain Design

Uses Bode plots of the **open-loop** function $L(s) = C(s)G(s)H(s)$ to predict **closed-loop** behavior.

- **Gain Margin (GM):** How much gain can be increased before instability.
- **Phase Margin (PM):** How much phase shift can be added before instability. **Directly related to damping and overshoot.**

Design Goal: Shape the Bode plot of $L(s)$ using $C(s)$ to achieve desired GM, PM, and bandwidth.

3.5 Controller Types

- **Proportional (P):** $C(s) = K_p$
  Effect: Reduces rise time, reduces but does not eliminate steady-state error.

- **Proportional-Integral (PI):** $C(s) = K_p + \frac{K_i}{s}$
  Effect: Eliminates steady-state error to a step input, but can worsen transient response.

- **Proportional-Derivative (PD):** $C(s) = K_p + K_d s$
  Effect: Increases stability, reduces overshoot, improves transient response.

- **PID (Proportional-Integral-Derivative):** $C(s) = K_p + \frac{K_i}{s} + K_d s$
  The "workhorse" of industrial control.

Part 4: The Design Flowchart

1. **Model the Plant:** Derive $G(s)$ from physics or identify it experimentally.

2. **Analyze the Plant:** Check its stability, step response, and frequency response.

3. **Define Specifications:** e.g., "Overshoot ¡ 5%", "Settling time = 2s", "Zero steady-state error".

4. **Choose a Control Strategy:** Start simple (P, PI, PID). Use:
   - **Root Locus** to place closed-loop poles.
   - **Frequency Response** to achieve target Phase/Gain Margin.

5. **Analyze the Closed-Loop System:** Simulate. Do you meet all specs?

6. **Iterate:** If not, modify your controller and go back to Step 4.

Key Takeaways & Mnemonics

- **LTI is Linear & Time-Invariant:** Superposition holds; parameters are constant.

- **Poles Dictate Stability:** LHP = Stable. RHP = Unstable. Imaginary Axis = Marginally Stable.

- **Feedback is Your Friend:** It lets you change a system's natural dynamics.

- **The Trade-Off Trio:** Control design is a balance between:

- **Speed of Response** (Rise Time)
- **Overshoot & Stability** (Damping)
- **Steady-State Error** and **Disturbance Rejection**